Rayleigh-Taylor Instability of a Stratified Magnetized Medium in the Presence of Suspended Particles

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The hydromagnetric Rayleigh-Taylor instability of a composite medium has been studied in the presence of suspended particles for an exponentially varying density distribution. The prevalent horizontal magnetic field and viscosity of the medium are assumed to be variable. The dispersion relation is derived for such a medium. It is found that the stability criterion is independent of both viscosity and suspended particles. The system can be stabilized for an appropriate value of the magnetic field. It is found that the suspended particles can suppress as well as enhance the growth rate of the instability in certain regions. The growth rates are obtained for a viscid medium with the inclusion of suspended particles and without it. It has been shown analytically that the growth rate is modified by the inclusion of the relaxation frequency parameter of the suspended particles.

I. Introduction

The problem of the Rayleigh-Taylor instability of a heavier fluid supported by a lighter one has been widely discussed in recent years due to its relevance in controlled thermonuclear experiments, inertial fusion, magnetohydrodynamic power generation and astrophysics. The magnetic field generally has a stabilizing effect on the flow of a conducting fluid in Rayleigh-Taylor configuration. Chandrasekhar [1] has presented a detailed and comprehensive account of the investigations carried out under various assumptions of hydrodynamics and hydromagnetics. He has discussed the problem of Rayleigh-Taylor instability of a heavier fluid supported against gravity by a lighter fluid considering both infinitely conducting fluids incompressible and permeated by a uniform horizontal magnetic field. Further, the [1] studied the influence of one dimensional density stratification on the stability of the plasma as a separate case. Bhatia [2] extended the problem to include the effect of collisional frequency of neutrals on the growth rate of Rayleigh-Taylor instability of a magneto-fluid with stratified density and observed its stabilizing influence. Bhatia [3] also examined the case of two superposed fluids of variable viscosity and infinite conductivity immersed in a uniform horizontal magnetic field and found that the growth rate was suppressed on account of viscosity. Kent [4] has studied the effect of a variable horizontal magnetic field on the stability of parallel flows and found that the configuration is unstable under specific conditions. Recently Sharma and Thakur [5] have investigated the Rayleigh-Taylor instability of a partially ionized viscous medium in the presence of a variable horizontal magnetic field in which there is a one dimensional density gradient along the vertical direction.

In addition to this, the problem of dusty gas in MHD is discussed at length by various authors. Scanlon and Segel [6] have examined the effect of suspended particles in hydromagnetics in connection with Benard convection and found that the critical Rayleigh number is reduced. Chhajlani et al. [7] have incorporated the self gravitation in rotating magnetized gas particle media with suspended particles. Sharma and Sharma [8] have also analyzed the Rayleigh-Taylor instability for a medium consisting of two superposed fluids including suspended particles and showed that the criteria determining stability and instability of the system remain uninfluenced by the presence of suspended particles.

In view of the importance of suspended particles, we have carried out a Rayleigh-Taylor stability analysis of a composite MHD medium with suspended particles. Sharma and Sharma [8] have considered the problem of Rayleigh-Taylor instability of two fluids with unequal densities, but we wish to investigate the problem with a single fluid having a vertical density gradient. Thus our object is to

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discuss the Rayleigh-Taylor instability of a magnetofluid of stratified density in the presence of suspended particles. We regard the horizontal magnetic field and the viscosity to be variable. We formulate the problem for the general case and discuss both the longitudinal and transverse modes of propagation. We also investigate analytically the growth rate of the instability with the relaxation frequency of the suspended particles.

II. Formulation of the Problem

Consider a horizontal gas-particle layer of depth d, unbounded in the horizontal (x, y) directions and bounded by surfaces at $z = 0$ and $z = d$. The fluid is considered infinitely conducting and viscid. The particles are assumed to be nonconducting. The system is embedded in a variable horizontal magnetic field $(H(z), 0, 0)$ acting transverse to the direction of gravity $\boldsymbol{g}$ $(0, 0, g)$.

Let $\boldsymbol{v}(\boldsymbol{x}, t)$ and $N(\boldsymbol{x}, t)$ describe the velocity field and the number density of the particles. Assuming the particles to be of uniform size and spherical shape, the net effect of the particles on the gas is equivalent to an extra body force term per unit volume $KN(\boldsymbol{v}-\boldsymbol{u})$. Here $\boldsymbol{u}$ (v, v, w) represents the velocity field of the gas, $K = 6\pi\varrho\nu\alpha$ (Stoke's drag formula) where ν is the kinematic viscosity and α the particle radius. The relevant basic equations of the hydromagnetic fluid-particle medium are

$$\left[\varrho\frac{\partial \boldsymbol{u}}{\partial t} + \varrho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] = -\nabla p + \frac{\mu_e}{4\pi}[(\nabla\times\boldsymbol{h})\times\boldsymbol{H}] + \mu\nabla^2\boldsymbol{u} + \boldsymbol{g}\varrho + \left(\frac{\partial w}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{u}}{\partial z}\right)\frac{d\mu}{dz} + KN(\boldsymbol{v}-\boldsymbol{u}), \quad (1)$$

$$\frac{\partial \boldsymbol{H}}{\partial t} = (\boldsymbol{H}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{H}, \quad (2)$$

$$\nabla\cdot\boldsymbol{u} = 0, \quad \nabla\cdot\boldsymbol{H} = 0, \quad (3)$$

$$\frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\boldsymbol{u}) = 0, \quad (4)$$

where ϱ, p, μ denote the density, the pressure and the magnetic permeability of the gas, respectively, $\boldsymbol{x} = (x, y, z)$. The density changes are small except in the gravity term. The force exerted by the gas on the particles is equal and opposite to that exerted by the particles on the gas. The buoyancy force on the particles is neglected as its stabilizing effect for two free boundaries is extremely small. Interparticle distances are assumed to be so large that interparticle reactions can be ignored (Scanlon and Segel [6]). Under these restrictions the equations of motion and continuity for the particles are

$$mN\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -mN\boldsymbol{g} + KN(\boldsymbol{v}-\boldsymbol{u}) \quad (5)$$

$$\frac{\partial N}{\partial t} + \nabla\cdot(N\boldsymbol{v}) = 0, \quad (6)$$

where mN is the mass of the particles per unit volume. The initial state is characterized by $\boldsymbol{u}_0 = 0$, $v_0 = 0$ and $N_0 =$ constant.

III. Perturbed Equations

The stability of the initial state is studied by writing the solution to the full equations as the initial state plus a perturbation term (denoted by primes):

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}', \quad \boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}', \quad N = N_0 + N',$$
$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{h}', \quad p = p_0 + \delta p \quad \text{and} \quad \varrho = \varrho_0 + \delta\varrho,$$

where δp and $\delta\varrho$ denote the perturbations in the pressure p and density ϱ, respectively. Upon assuming infinitesimal disturbances and dropping primes, we find that the linearized equations for the perturbed physical quantities becomes

$$\varrho\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\delta p + \frac{\mu_e}{4\pi}(\nabla\times\boldsymbol{h})\times\boldsymbol{H} + \frac{\mu_e}{4\pi}(\nabla\times\boldsymbol{H})\times\boldsymbol{h} + \boldsymbol{g}\,\delta\varrho + \left(\frac{\partial w}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{u}}{\partial z}\right)\frac{d\mu}{dz} + KN(\boldsymbol{v}-\boldsymbol{u}), \quad (7)$$

$$\left(\tau\frac{\partial}{\partial t} + 1\right)\boldsymbol{v} = \boldsymbol{u}, \quad (8)$$

$$\partial\boldsymbol{h}/\partial t = (\boldsymbol{H}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{H}, \quad (9)$$

$$\nabla\cdot\boldsymbol{u} = 0, \quad \nabla\cdot\boldsymbol{h} = 0, \quad (10)$$

$$\frac{\partial}{\partial t}\delta\varrho + (\boldsymbol{u}\cdot\nabla)\varrho = 0, \quad (11)$$

where $\tau = m/K$ denotes the relaxation time which governs the effect of particles.

Analyzing the disturbance in terms of normal modes we seek solutions of the above equations in which the perturbation have a space and time dependence of the form

$$F(z) = i k_x x + i k_y y + n t, \quad (12)$$

where $k = \sqrt{k_x^2 + k_y^2}$ is the wave number and n is the growth rate of the harmonic disturbance.

Substitution of (12) in (8) gives $\boldsymbol{v}$ in terms of $\boldsymbol{u}$. Further, inserting $\boldsymbol{v}$ in (7) and employing (12) we obtain the components of (7) as

$$[\varrho(\tau n+1)+mN]\,n\,u = -(1+\tau n)\,i k_x\,\delta p + \mu(1+\tau n)(D^2-k^2)\,u + (i k_x w + D u)(1+\tau n)\,D\mu + \frac{\mu_e}{4\pi}(1+\tau n)(DH)\,h_z\,, \tag{13}$$

$$[\varrho(\tau n+1)+mN]\,n\,v = -(1+\tau n)\,i k_y\,\delta p + \mu(1+\tau n)(D^2-k^2)\,v + (1+\tau n)(i k_y w + Dv)\,D\mu + \frac{\mu_e}{4\pi}(1+\tau n)\,H\,(i k_x h_y - i k_y h_x)\,. \tag{14}$$

$$[\varrho(\tau n+1)+mN]\,n\,w = -(1+\tau n)\,D\,\delta p + \frac{\mu_e}{4\pi}H(1+\tau n)(i k_x h_z - D h_x) + \mu(1+\tau n)(D^2-k^2)\,w - \frac{\mu_e}{4\pi}(1+\tau n)(DH)\,h_x + 2(1+\tau n)\,Dw\,D\mu + \frac{g}{n}(1+\tau n)(D\varrho)\,w\,, \tag{15}$$

where D stands for the operator $\frac{d}{dz}$.

Also we have, from (3)–(5),

$$n h_x = i k_x H u - w DH\,, \tag{16}$$

$$n h_y = i k_x H v\,, \tag{17}$$

$$n h_z = i k_x H w\,, \tag{18}$$

$$n\,\delta\varrho = -w D\varrho, \tag{19}$$

$$i k_x u + i k_y v + Dw = 0\,, \tag{20}$$

$$i k_x h_x + i k_y h_y + D h_z = 0\,. \tag{21}$$

Multiplying (7) and (8) by $i k_x$ and $i k_y$, respectively, and after adding using (9) to eliminate δp we obtain

$$n[(\tau n+1)+mN/\varrho][D(\varrho D w) - k^2\varrho w] - \frac{\mu_e}{4\pi} i k_x(\tau n+1)\cdot[H(D^2-k^2)\,h_z - (D^2 H)\,h_z] - \mu(D^2-k^2)^2 w(\tau n+1) - (\tau n+1)(D^2-k^2)\,D\mu\,Dw - (\tau n+1)\,D^2\mu\,(D^2+k^2)\,w + \frac{g k^2}{n}(\tau n+1)(D\varrho)\,w = 0\,. \tag{22}$$

Finally substituting for h_z from (13) we get the following differential equation in w:

$$n[(\tau n+1)+mN/\varrho][D(\varrho D w) - k^2\varrho w] - \mu(\tau n+1)(D^2-k^2)^2 w - (\tau n+1)(D^2-k^2)\,D\mu\,Dw - (\tau n+1)\,D^2\mu\,(D^2+k^2)\,w + \frac{\mu_e}{4\pi n}k_x^2(1+\tau n)\{H^2(D^2-k^2)\,w + D(H^2)\,Dw\} + \frac{g k^2}{n}(\tau n+1)(D\varrho)\,w = 0\,. \tag{23}$$

IV. Dispersion Relation and Discussion

We shall now solve (23) for a continuously stratified (one dimensionally) composite medium of finite depth d in which the density of the magnetized pure gas is given by

$$\varrho(z) = \varrho_0 \exp(\beta z), \quad 0 \leqq z \leqq d; \qquad (24)$$

$$\varrho(z) = 0 \text{ elsewhere}.$$

where β is a constant and ϱ_0 is the density at the lower boundary. It should be pointed out here that a similar stratification is assumed for the suspended particles. For the sake of mathematical simplicity, we consider the distribution of coefficient of viscosity and magnetic field also to be exponentially stratified as

$$\left.\begin{aligned} \mu(z) &= \mu_0 \exp(\beta z) \\ H^2(z) &= H_0^2 \exp(\beta z) \end{aligned}\right\}. \qquad (25)$$

It follows from the above equation that the kinematic viscosity and the Alfven velocity $V_A = (H^2 \mu_e / 4\pi \varrho_0)^{1/2}$ are constant throughout the medium.

IV. (A) Inviscid Medium Without Suspended Particles

It is elucidating and easier to reduce (23) first for the inviscid and magnetized pure gas medium (i.e. in the absence of suspended particles). By employing (24) and (25) in (23) for the case IV. A we obtain after simplification:

$$D^2 w + \beta D w + \frac{(g\beta/n - k_x^2 V_A^2/n - n)\, k^2 w}{(n + k_x^2 V_A^2/n)} = 0. \qquad (26)$$

The solution of (26) appropriate to the boundary condition at $z = 0$, $w = 0$ can be written as

$$w(z) = A(\exp(m_1 z) - \exp(m_2 z)), \qquad (27)$$

where m_1 and m_2 are the roots of the equation

$$m^2 + \beta m + \frac{[g\beta/n - k_x^2 V_A^2/n - n]\, k^2}{(n + k_x^2 V_A^2/n)} = 0. \qquad (28)$$

The vanishing of w at $z = d$ requires

$$\exp(m_1 - m_2)\, d = 1, \qquad (29)$$

i.e.

$$(m_1 - m_2)\, d = 2 i s \pi,$$

where s is an integer.

From (28) and (29) the dispersion relation can be worked out as

$$n^2 + [k_x^2 V_A^2 - g\beta k^2/(l^2 + k^2)] = 0, \qquad (30)$$

where we have written

$$l^2 = \pi^2 s^2/d^2 + \beta^2/4.$$

Here we can distinguish two modes of propagation

(i) Longitudinal mode ($k_x = k$, $k_y = 0$). The dispersion relation in this case is (30) with $k_x = k$. If $\beta < 0$, we find that the configuration is always stable.

However when $\beta > 0$ we find that the medium becomes stable only for wave numbers which satisfy the inequality

$$k^2 > g\beta/V_A^2 - l^2. \qquad (31)$$

The above condition is the same as was obtained by Talwar [9].

(ii) Transverse mode ($k_x = 0$, $k_y = k$). In this case (30) reduces to

$$n^2 - \frac{g\beta k^2}{(l^2 + k^2)} = 0. \qquad (32)$$

Therefore, the medium is stable or unstable according as $\beta \gtrless 0$, as discussed by Chandrasekhar [1]. Hence, we could extract the dispersion relation (30) as a special case of (23) and the results are the same as discussed by the previous authors.

IV.(B) Viscid Medium Including Suspended Particles

In this section we will restrict our treatment to the case when both bounding surfaces are free. This is somewhat unrealistic but nevertheless of importance because it allows for an exact solution of the problem. Further, this analysis is relevant in the case of stellar atmospheres as pointed out by Spiegel [10]. We shall now derive the dispersion relation from (23) which includes both the magnetized gas and suspended particles. Substituting (24) and (25) in (23) and neglecting the effect of heterogeneity on inertia, we have

$$(D^2 - k^2)^2 w - \left[\frac{k_x^2 V_A^2}{n \nu_0} + \frac{n(\tau n + 1 + mN/\varrho)}{(\tau n + 1)\, \nu_0}\right] \cdot (D^2 - k^2) w - \frac{g\beta k^2}{n \nu_0} w = 0. \qquad (33)$$

In a study of thermal instability in hydromagnetics Bhatia [11] has asserted that the case of rigid boundaries can be solved by a variational method, but the results of free and rigid boundaries differ very little. Hence it is justified to solve this problem for two free boundaries. The relevant boundary conditions, on following Bhatia [11] and Shrivastava [12] for the case of free-free boundaries, are

$$w = 0\,, \quad D^2 w = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = d\,, \tag{34}$$

where $\nu_0 = \mu_0/\varrho_0$. The proper solution of (33) in view of the boundary conditions stated in (34) is

$$w = A \sin(m' \pi z/d)\,, \tag{35}$$

where m' is an integer.

In fact w and all its even derivatives vanish at $x = 0$, d. We substitute the value of w in (33), which on further simplification yields

$$\left[\left(\frac{m'\pi}{d}\right)^2 + k^2\right]^2 + \left\{\frac{k_x^2 V_A^2}{n\nu_0} + \frac{n(\tau n + 1 + mN/\varrho)}{(\tau n + 1)\nu_0}\right\}\left[\left(\frac{m'\pi}{d}\right)^2 + k^2\right] - g\beta k^2/n\nu_0 = 0\,. \tag{36}$$

The above equation can be written as

$$\tau n^3 + n^2[mN/\varrho + L\nu_0\tau] + n[L\nu_0 + \tau(k_x^2 V_A^2 - g\beta k^2/L)] + [k_x^2 V_A^2 - g\beta k^2/L] = 0\,, \tag{37}$$

where we have put $(m'\pi/d)^2 + k^2 = L$.

Next, writing $f_s = 1/\tau$ in (37), where f_s is the relaxation frequency parameter of the suspended particles, we obtain

$$n^3 + n^2[f_s(1 + mN/\varrho) + L\nu_0] + n[L\nu_0 f_s + (k_x^2 V_A^2 - g\beta k^2/L)] + f_s[k_x^2 V_A^2 - g\beta k^2/L] = 0\,. \tag{38}$$

It is readily seen from (38) that for $\beta < 0$, which is the criterion of stable density stratification, the above equation does not admit any positive root, implying thereby that the system is stable. For unstable stratification $\beta > 0$. The gas-particle medium is stable or unstable according to

$$k_x^2 V_A^2 \gtrless g\beta k^2/L\,. \tag{39}$$

Hence, it is evident from (38) and (39) that the stability criterion is independent of the presence of suspended particles and viscosity.

From (39) we find that for $\beta > 0$ the system is unstable in the absence of a magnetic field. However, the system can be stabilized for a value of the magnetic field which satisfies the inequality

$$V_A^2 > g\beta k^2/k_x^2 L\,. \tag{40}$$

We find that for $\beta > 0$ and $k_x^2 V_A^2 < g\beta k^2/L$ (38) has, at least, one positive root which will destabilize the medium for all wave numbers. Let n_0 denote the positive root of (38), hence

$$n_0^3 + n_0^2[f_s(mN/\varrho + 1) + L\nu_0] + n_0[L\nu_0 f_s + (k_x^2 V_A^2 - g\beta k^2/L)] + f_s[k_x^2 V_A^2 - g\beta k^2/L] = 0\,. \tag{41}$$

In order to have an insight into the role of suspended particles and viscosity on the growth rate of unstable modes, we evaluate dn_0/df_s and $dn_0/d\nu_0$ and examine their nature. From (41) we obtain

$$\frac{dn_0}{df_s} = -\frac{n_0^2(mN/\varrho + 1) + n_0 L\nu_0 + (k_x^2 V_A^2 - g\beta k^2/L)}{3n_0^2 + 2n_0[f_s(1 + mN/\varrho) + \nu_0 L] + [k_x^2 V_A^2 - g\beta k^2/L + L f_s \nu_0]}\,. \tag{42}$$

Therefore, in addition to $k^2 > k_x^2 V_A^2 L/g\beta$, which makes the medium unstable, we find that if either

$$|(k_x^2 V_A^2 - g\beta k^2/L)| > \{3n_0^2 + 2n_0[f_s(1 + mN/\varrho) + \nu_0 L] + L f_s \nu_0\} \tag{43}$$

or

$$|(k_x^2 V_A^2 - g\beta k^2/L)| < [n_0^2(1 + mN/\varrho) + n_0 L\nu_0]\,, \tag{44}$$

dn_0/df_s is always negative. In writing (43) and (44) we have taken note of the fact that mN/ϱ can not exceed 1.

Thus, the growth rate decreases with increasing relaxation frequency of the suspended particles. However, the growth rate is enhanced with increase in the relaxation frequency parameter if

$$[n_0^2(1+mN/\varrho)+n_0 L\nu_0] < |(k_x^2 V_A^2 - g\beta k^2/L)| < \{3n_0^2 + 2n_0[f_s(1+mN/\varrho)+\nu_0 L + L f_s \nu_0\}, \quad (45)$$

because in this case dn_0/df_s becomes positive. Equation (45) defines the region where suspended particles have destabilizing influence.

Next, from (41) we obtain

$$\frac{dn_0}{d\nu_0} = -\frac{-L n_0(n_0+f_s)}{3n_0^2+2n_0[f_s(1+mN/\varrho)+L\nu_0]+[k_x^2 V_A^2 - g\beta k^2/L + L f_s \nu_0]}. \quad (46)$$

It is evident from (46) that an increase in kinematic viscosity results in decreasing the growth rate of the disturbance if

$$|(k_x^2 V_A^2 - g\beta k^2/L)| > \{3n_0^2 + 2n_0[f_s(1+mN/\varrho)+L\nu_0] + L f_s \nu_0\}, \quad (47\,a)$$

and the growth rate of the instability increases with increasing kinematic viscosity if

$$|(k_x^2 V_A^2 - g\beta k^2/L)| < \{3n_0^2 + 2n_0[f_s(1+mN/\varrho)+L\nu_0] + L f_s \nu_0\}. \quad (47\,b)$$

Taking (37) and following the same procedure as above we can deduce the growth rate with viscosity in the absence of suspended particles i.e. ($\tau = 0$) which is

$$\frac{dn_0}{d\nu_0} = -\frac{L n_0}{2n_0 + L\nu_0}. \quad (48)$$

From (48) we conclude that the growth rate is always negative. Comparison of (46) and (48) reveals an interesting feature in that the simultaneous presence of suspended particles and viscosity can increase as well as decrease the growth rate, whereas if suspended particles are not included, then the viscosity always reduces the growth rate of the instability. Similar consequence of the viscosity have been reported in [3] but in the context of hydromagnetic Rayleigh-Taylor instability of two superposed fluids.

Thus we have studied Rayleigh-Taylor instability of a magnetized medium including suspended particles considering the viscosity, density and magnetic field to be variable. The stability criterion is independent of the suspended particles and viscosity. The relaxation frequency of the suspended particles has a stabilizing influence on the growth rate in the regions (43, 44), and it has a destabilizing effect i.e. enhances the growth rate in the region (45). The kinematic viscosity is found to have a stabilizing as well as a destabilizing influence on the growth rate of instability under the conditions expressed in (47 a) and (47 b), respectively. But it is found that the viscosity reduces the growth rate of a clean gas medium (without suspended particles).

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